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On the zeros of 3j coefficients: polynomial degree versus recurrence order

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Abstract. For $3j$ coefficients of SU(2) two quantities, the degree n and the order m, are defined. The first, the (polynomial) degree n, is well known as the number of terms in the polynomial part of the single-sum expression of the $3j$ coefficient minus one. The second, the (recurrence) order m, is new. From an exhaustive computer search for non-trivial zeros of $3j$ coefficients, it follows that both quantities play an important role in classifying the zeros. Explicit formulae for all zeros of order 1 are obtained and several new infinite sequences of zeros related to Pell equations are presented.

1. Introduction

For the $3j$ and $6j$ coefficients of SU(2), there exists a class of zeros which have been called 'non-trivial' or 'structural' zeros as opposed to the 'trivial' zeros resulting from a symmetry ($3j$ coefficient) or due to a violation of one or more triangle conditions ($3j$ and $6j$ coefficient). In the encyclopedia volume *The Racah–Wigner Algebra in Quantum Theory* (Biedenharn and Louck 1981), this subject appears as Special Topic 10. Some non-trivial zeros play a role in physics: the fact that the $6j$ coefficient

$$\begin{Bmatrix} 2 & 2 & 2 \\ 3/2 & 3/2 & 3/2 \end{Bmatrix}$$

is zero implies that quadrupole radiation from an aligned state having $j = 3/2$ to a ground state $j = 3/2$ is isotropic; similarly, the zeros of $3j$ coefficients of the form

$$\begin{pmatrix} L & L & 2\nu \\ 1 & -1 & 0 \end{pmatrix}$$

have been important in the investigation of whether competing radiations exhibit different angular distributions (Arfken *et al* 1951). These examples motivated the systematic study of non-trivial zeros (Biedenharn and Louck 1981).

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Apart from this, there is also a mathematical motivation, since non-trivial zeros of $3j$ and $6j$ coefficients correspond to roots of special polynomials. Moreover, non-trivial zeros have appeared in relation to exceptional Lie groups and algebras (Koozekanani and Biedenharn 1974, Biedenharn and Louck 1981, Van der Jeugt *et al* 1983, De Meyer *et al* 1984, Van den Berghe *et al* 1984, Van der Jeugt 1992), which seems to be another surprising relationship stimulating further research.

Some studies in this area have been numerical in approach. Bowick (1976) shortened the tables of zeros for $3j$ coefficients (Varshalovich *et al* 1975) and $6j$ coefficients (Koozekanani and Biedenharn 1974), taking into account the symmetries discovered by Regge (1958, 1959). Here, all the symmetries will be taken into account.

In a systematic approach, the non-trivial zeros of $3j$ and $6j$ coefficients can be classified by the minimum length of the single-sum expression for the coefficients. This minimum length corresponds to the number of terms when the coefficient is rearranged as a generalized hypergeometric series (Lindner 1985, Srinivasa Rao 1985, Srinivasa Rao and Rajeswari 1985). The number of terms in this sum minus one will be called the *degree* of the coefficient (in some papers it is called the weight). Thus zeros of degree n ($n > 0$) are, by definition, non-trivial zeros. The zeros of degree 1 are quite easy to find for the $3j$ coefficients and explicit expressions for them, not taking Regge symmetries into account, have been given by Varshalovich *et al* (1975). The zeros of degree 1 of the $6j$ coefficients have been studied by Brudno (1985), Brudno and Louck (1985), Bremner and Brudno (1986), Srinivasa Rao and Rajeswari (1987), and Srinivasa Rao *et al* (1988); here the most general parametrization requires a constraint equation amongst the parameters. Using the Pell equation, Beyer *et al* (1986) showed that there are infinite sequences of zeros of degree 2 for $6j$ coefficients. Louck and Stein (1987) obtained the same result for $3j$ coefficients. Algorithms for these two cases were obtained by Srinivasa Rao and Chiu (1989). For the zeros of $6j$ coefficients of higher degree, Brudno (1987) found special cases where the Pell equation can be used to generate them.

The original motivation for the present paper was an investigation of the distribution of the zeros of $3j$ coefficients with respect to their degree n and the sum J of the angular momentum quantum numbers involved (this sum is invariant for Regge symmetries), by means of a greatly extended computer search. For low angular momenta, we originally used integer arithmetic developed to compute coefficients of point harmonics for the cubic group by Conte and Raynal (1985) and for the icosahedral group by Raynal (1985). Such calculations are very slow and also limited in size by the fact that only prime numbers smaller than 200 are used to represent integers. We realized that such a slow technique is not necessary: the polynomial part or series expression of these coefficients is an alternating sum, and the result obtained with floating-point operations divided by the sum of the absolute values of the terms in the series must be smaller than the precision of the computer if the coefficient vanishes. Candidate zeros were first selected using the usual double precision; then quadruple precision was used to verify them. With quadruple precision, calculations are so accurate that no doubt exists about whether a candidate zero is identically zero or whether it is just a very small number, in the domain which has been scanned here. The thresholds were generally set to 10^{-14} for double precision and 10^{-28} for quadruple precision on a Convex computer.

The first computer search for the zeros of $3j$ coefficients with $J \leqslant 240$ gave many zeros of high degree. However, inspection showed that most of these can be written with small magnetic quantum numbers. This observation was the starting point for this work, indicating that apart from the *degree* n a new quantity, the *order* m, can be introduced to classify the zeros of $3j$ coefficients.

In section 2, the notation for $3j$ coefficients is summarized. Some sets of $3j$ coefficients which never vanish are discussed in section 3. In our terminology, these are $3j$ coefficients of order 0. In section 4, we show how recurrence relations can give zeros near the coefficients of order 0 and define the order m of a zero. In section 5, the equations for zeros of order 1 are solved and these are given explicitly. Section 6 presents partial results for zeros of order 2 and 3, and in section 7 some further results are deduced by inspecting explicit computer lists of zeros. In particular, some new infinite sequences of zeros are presented. For $6j$ coefficients, a similar search for the zeros has been made and this will be the subject of a following paper.

2. Expressions for the 3j coefficients

The $3j$ coefficient of SU(2) is conveniently expressed as a generalized hypergeometric series ${}_3F_2[1]$:

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = C\,{}_3F_2[-a-b+c, -b-\beta, -a+\alpha; -a+c-\beta+1, -b+c+\alpha+1; 1] \tag{1}$$

where C is some non-zero factor. Whipple (1925) studied the different aspects of such generalized hypergeometric series by introducing six parameters r_i ($i = 0, 1, \ldots, 5$) the sum of which is zero. All the symmetries of the $3j$ coefficients can be deduced from his results and a complete list of the ${}_3F_2[1]$ applied to this problem was given by Raynal (1978). However, for our purpose, we shall use the 3×3 square symbol introduced by Regge (1958):

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{Vmatrix} -a+b+c & a-b+c & a+b-c \\ a+\alpha & b+\beta & c+\gamma \\ a-\alpha & b-\beta & c-\gamma \end{Vmatrix} \tag{2}$$

in which the sum in any row or column is $J = a + b + c$. The transposition and any permutation of rows or columns correspond to the same value of the $3j$ coefficient, up to a sign. Thus the $3j$ coefficient possesses 72 symmetries. The smallest integer n in this symbol defines the 'degree' of the $3j$ coefficient and the number of terms of the ${}_3F_2[1]$ in (1) will be $n + 1$.

In the process of listing all the $3j$ coefficients of degree n for a given value of J, it is useful to avoid duplication of coefficients which are related by any of the 72 symmetries. For this purpose, we shall only consider Regge symbols of the following form:

$$\begin{Vmatrix} J-x-z & J-y-t & n \\ x & y & J-x-y \\ z & t & J-z-t \end{Vmatrix} = C'\,{}_3F_2[-n, -y, -z; x-n+1, t-n+1; 1] \tag{3}$$

with

$$x+y+z+t = J+n \qquad x \geqslant y \geqslant z \geqslant n \qquad x \geqslant t \geqslant n \qquad (z \geqslant t \text{ if } x = y). \tag{4}$$

Using the Regge transformations, every symbol can be written in this standard form. Bowick (1976) used the parameters $[x, t, n, y-n, z-n]$ introduced by Bryant and Jahn (1960) to define independent $3j$ coefficients, with similar conditions.

Using the notation

$$(x)_{-i} = x(x-1)\dots(x-i+1)$$

we obtain the symmetric expression

$${}_3F_2[-n,-y,-z;x-n+1,t-n+1;1] = C''\sum_{i=0}^{n}(-)^i\frac{n!}{i!(n-i)!}(x)_{i-n}(y)_{-i}(z)_{-i}(t)_{i-n}. \tag{5}$$

The expression for a $3j$ coefficient in terms of a ${}_3F_2[1]$ series is not unique. The same $3j$ coefficient can be written by means of different ${}_3F_2[1]$ series, all involving at least the same number of terms. Raynal (1978) has summarized all these expressions. One of the nine formulae in Raynal (1978) or one of the seven formulae listed by Varshalovich *et al* (1975) is of special interest in the search for zeros. It has previously been obtained by Bandzaitis and Yutsis (1964) and reads here as

$${}_3F_2[-n,-z,-J-1;-z-x,-z-t;1].$$

When rewriting this as an alternating sum of integers as above, one can see that $J+1$ is a factor in all these integers, except the first. Therefore, if $J+1$ is a prime number, the sum cannot vanish. This result was also obtained by Bryant and Jahn (1960).

3. Non-zero 3*j* coefficients

Motivated by the fact that many of the non-trivial $3j$ coefficients of high degree n seem to have an expression with small magnetic quantum numbers, we approached the study of the zeros from a different viewpoint. In this section we list four sets of $3j$ coefficients which never vanish; they will be called $3j$ coefficients of order 0. In the following section we shall see how non-trivial zeros of order $m>0$ can be related to these non-vanishing $3j$ coefficients of order 0.

When $n=0$, the $3j$ coefficient reduces to a non-zero constant. But the expression for the $3j$ coefficient also reduces to a single term when $x=z$ and $y=t$ (the two last rows of the Regge symbol are then equal) because the generalized hypergeometric series can be summed with Dixon's theorem (Dixon 1903). Two other theorems due to Whipple and Watson, quoted by Erdelyi (1953), give the same result when applied to other expressions for the $3j$ coefficient. Using Dixon's theorem (cf (4.4.5) in Erdelyi 1953) we get the result:

$$\begin{aligned}&{}_3F_2[-n,-x,-y;1-n+x,1-n+y;1]\\&\qquad=\Gamma\begin{pmatrix}1-\frac{1}{2}n,1-\frac{1}{2}n+x+y,1-n+x,1-n+y\\1-n,1-n+x+y,1-\frac{1}{2}n+x,1-\frac{1}{2}n+y\end{pmatrix}.\end{aligned} \tag{6}$$

With

$$a=x \qquad b=y \qquad c=x+y-n \qquad 2p=a+b+c=J$$

one gets, after some simplification, the well known result for the 'parity' $3j$ coefficient:

$$\begin{pmatrix}a&b&c\\0&0&0\end{pmatrix} = (-)^p\left[\frac{(2p-2a)!(2p-2b)!(2p-2c)!}{(2p+1)!}\right]^{1/2}\frac{p!}{(p-a)!(p-b)!(p-c)!} \tag{7}$$

if J is even, and

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = 0 \tag{8}$$

if J is odd. So, if two rows or two columns of the Regge symbol are identical and if J is odd, the $3j$ coefficient vanishes: it is a 'trivial' zero. In contrast, if J is even, the $3j$ coefficient cannot vanish.

There are other sets of non-zero coefficients which can be deduced from this one by means of recurrence relations. Among them are the $3j$ coefficients with magnetic quantum numbers $\pm\frac{1}{2}$ used to obtain matrix elements for particles of spin $\frac{1}{2}$ in the helicity formalism (Raynal 1967). Using, for example, the relation (A7) in Raynal (1979), one can deduce that (for b integer, a and c half-integers):

$$\{(2a+1)(2c+1)\}^{1/2}\begin{pmatrix} a & b & c \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \{(-a+b+c+1)(a+b-c)\}^{1/2}\begin{pmatrix} a-\frac{1}{2} & b & c+\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$
$$+\{(a+b+c+2)(a-b+c+1)\}^{1/2}\begin{pmatrix} a+\frac{1}{2} & b & c+\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = 0. \tag{9}$$

Again with $a+b+c = J = 2p$, we get

$$\begin{pmatrix} a & b & c \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = (-1)^{p+1}\left[\frac{(2p-2a)!(2p-2b)!(2p-2c)!}{(2a+1)(2c+1)(2p+1)!}\right]^{1/2}$$
$$\times \frac{2\quad p!}{(p-a-\frac{1}{2})!(p-b)!(p-c-\frac{1}{2})!} \tag{10}$$

if J is even, and

$$\begin{pmatrix} a & b & c \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = (-1)^{p+3/2}\left[\frac{(2p-2a)!(2p-2b)!(2p-2c)!}{(2a+1)(2c+1)(2p+1)!}\right]^{1/2}$$
$$\times \frac{2(p+\frac{1}{2})!}{(p-a)!(p-b-\frac{1}{2})!(p-c)!} \tag{11}$$

if J is odd. None of these coefficients can vanish.

Another set of non-zero $3j$ coefficients for a, b and c integer and $J = a+b+c$ odd can be obtained. From (A7) in Raynal (1979) one can also deduce that

$$(b-c)\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} = \{(a-b+c+1)(a+b-c)(b+1)c\}^{1/2}\begin{pmatrix} a & b-\frac{1}{2} & c+\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$
$$-\{(a+b-c+1)(a-b+c)b(c+1)\}^{1/2}\begin{pmatrix} a & b+\frac{1}{2} & c-\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{12}$$

This gives

$$\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} = (-1)^{p+1/2}\left[\frac{(2p-2a)!(2p-2b)!(2p-2c)!}{b(b+1)c(c+1)(2p+1)!}\right]^{1/2}$$
$$\times \frac{2(p+\frac{1}{2})!}{(p-a-\frac{1}{2})!(p-b-\frac{1}{2})!(p-c-\frac{1}{2})!} \tag{13}$$

valid only for $J = a + b + c$ odd.

The non-zero $3j$ coefficients of formulae (7), (10), (11) or (13) will be relevant in studying non-trivial zeros with small magnetic quantum numbers.

It should be noticed that the expressions given here are not new: e.g. (13) can be obtained from equation (1.52) in Rotenberg *et al* (1959) or from equation (3.7.15) in Edmonds (1960), who used this in his derivation of the parity $3j$ coefficient. More recently, Rashid (1986) rederived relation (13) using a complicated transformation between hypergeometric functions. Some other formulae in the following section (but not all) have also been obtained by Rashid.

4. The order m of a $3j$ coefficient

To define an order m, we consider recurrence relations between three 'contiguous' $3j$ coefficients as defined by Raynal (1978) (this is a generalization of the notion of contiguous hypergeometric functions). The three-term recurrence relation which will be used here is (Raynal 1979, part of equation (A7))

$$\begin{aligned}
&- S(a,b,c,\alpha,\beta,\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(a,b,\alpha,\beta)\begin{pmatrix} a & b & c \\ \alpha-1 & \beta+1 & \gamma \end{pmatrix} \\
&\quad = S(a,b,c,-\alpha,-\beta,\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(a,b,-\alpha,-\beta)\begin{pmatrix} a & b & c \\ \alpha+1 & \beta-1 & \gamma \end{pmatrix} \\
&\quad = -S(b,c,a,\beta,\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(b,c,\beta,\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta-1 & \gamma+1 \end{pmatrix} \\
&\quad = S(b,c,a,-\beta,-\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(b,c,-\beta,-\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta+1 & \gamma-1 \end{pmatrix} \\
&\quad = -S(c,a,b,\gamma,\alpha,\beta)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(c,a,\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha+1 & \beta & \gamma-1 \end{pmatrix} \\
&\quad = S(c,a,b,-\gamma,-\alpha,\beta)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(c,a,-\gamma,-\alpha)\begin{pmatrix} a & b & c \\ \alpha-1 & \beta & \gamma+1 \end{pmatrix}
\end{aligned} \tag{14}$$

where

$$S(a,b,c,\alpha,\beta,\gamma) = \tfrac{1}{2}(a(a+1) + b(b+1) - c(c+1)) + \alpha\beta + \tfrac{1}{3}(\alpha - \beta)$$

$$T(a,b,\alpha,\beta) = ((a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1))^{1/2}.$$

Now the notion of *order* will be introduced. The non-zero $3j$ coefficients of formulae (7), (10), (11) and (13) will be called of order $m = 0$. Using the recurrence relation (14) once, for chosen values of (α, β, γ), one obtains up to six sets of $3j$ coefficients which can be expressed in terms of those of order 0. These $3j$ coefficients will be called of (recurrence) order $m = 1$. They are the following.

(i) Setting $\alpha = \beta = \gamma = 0$ in (14), and using a symmetry for the $3j$ coefficient, we get, for even J,

$$\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} = \frac{a(a+1) - b(b+1) - c(c+1)}{2\{b(b+1)c(c+1)\}^{1/2}}\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \qquad (J \text{ even}). \tag{15}$$

(ii) Setting $(\alpha, \beta, \gamma) = (0, 1, -1)$ in (14) and using the symmetries of $3j$ coefficients and (8), we get two relations, both valid only for odd values of J. They are

$$\begin{pmatrix} a & b & c \\ 0 & 2 & -2 \end{pmatrix} = \frac{a(a+1) - b(b+1) - c(c+1) + 2}{\{(b-1)(b+2)(c-1)(c+2)\}^{1/2}} \begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} \qquad (J \text{ odd}) \tag{16}$$

$$\begin{pmatrix} a & b & c \\ 1 & 1 & -2 \end{pmatrix} = \frac{(b-a)(a+b+1)}{\{a(a+1)(c-1)(c+2)\}^{1/2}} \begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} \qquad (J \text{ odd}). \tag{17}$$

For (17) a zero can be found only for $a = b$ and it is a trivial zero, since J is odd.

(iii) Setting $(\alpha, \beta, \gamma) = (0, \frac{1}{2}, -\frac{1}{2})$ (and using symmetries and relabellings for a, b and c), we get five new relations. They are

$$\begin{pmatrix} a & b & c \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} = \frac{a(a+1) - b(b+1) - c(c+1) - (b+\frac{1}{2})(c+\frac{1}{2}) + \frac{1}{2}}{\{(b-\frac{1}{2})(b+\frac{3}{2})(c-\frac{1}{2})(c+\frac{3}{2})\}^{1/2}} \begin{pmatrix} a & b & c \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad (J \text{ even}) \tag{18}$$

$$\begin{pmatrix} a & b & c \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} = \frac{a(a+1) - b(b+1) - c(c+1) + (b+\frac{1}{2})(c+\frac{1}{2}) + \frac{1}{2}}{\{(b-\frac{1}{2})(b+\frac{3}{2})(c-\frac{1}{2})(c+\frac{3}{2})\}^{1/2}} \begin{pmatrix} a & b & c \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad (J \text{ odd}) \tag{19}$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & 1 & -\frac{3}{2} \end{pmatrix} = \frac{(a+\frac{1}{2})(a+c+1) - b(b+1)}{\{b(b+1)(c-\frac{1}{2})(c+\frac{3}{2})\}^{1/2}} \begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \qquad (J \text{ even}) \tag{20}$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & 1 & -\frac{3}{2} \end{pmatrix} = \frac{(a+\frac{1}{2})(a-c) - b(b+1)}{\{b(b+1)(c-\frac{1}{2})(c+\frac{3}{2})\}^{1/2}} \begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \qquad (J \text{ odd}) \tag{21}$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = -\frac{(c+\frac{1}{2}) + (-1)^J(a+\frac{1}{2})}{\{b(b+1)\}^{1/2}} \begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \qquad (J \text{ even or odd}). \tag{22}$$

For equation (22) a zero can be found only when $a = c$ and J is odd. However, that would be a trivial zero. This relation has been used by Raynal (1967) to express all the $3j$ coefficients which appear in the helicity formalism with formulae (10) and (11).

Table 1. The type and corresponding vanishing condition for a $3j$ coefficient $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ of order 1, with $J = a + b + c$ and $K = a(a+1) - b(b+1) - c(c+1)$, is given. Some further constraints are posed in order to have a unique classification of order 1 zeros. In the last two columns, the number of zeros of degree larger than 1 obtained for each relation up to $J = 300$ and $J = 3000$ is given.

Type	α	β	γ	J	Condition	Constraints	$\leqslant 300$	$\leqslant 3000$
I	0	1	-1	Even	$K = 0$	$a > b \geqslant c$	68	1144
II	0	2	-2	Odd	$K + 2 = 0$	$a > b \geqslant c$	61	1087
III	0	$\frac{3}{2}$	$-\frac{3}{2}$	Even	$K + \frac{1}{2} - (b+\frac{1}{2})(c+\frac{1}{2}) = 0$	$a > b \geqslant c$	50	934
IV	0	$\frac{3}{2}$	$-\frac{3}{2}$	Odd	$K + \frac{1}{2} + (b+\frac{1}{2})(c+\frac{1}{2}) = 0$	$b > a > c$ $(a, b) \neq (c+\frac{1}{2}, c+1)$	50	934
V	$\frac{1}{2}$	1	$-\frac{3}{2}$	Even	$(a+\frac{1}{2})(a+c+1) - b(b+1) = 0$	$a \neq c$	62	1211
VI	$\frac{1}{2}$	1	$-\frac{3}{2}$	Odd	$(a+\frac{1}{2})(a-c) - b(b+1) = 0$	$a \neq c+2$	115	2165

Clearly, equations (15), (16), (18)–(21) give the zeros of the $3j$ coefficients of (recurrence) order 1. The conditions for these cases, labelled by (I)–(VI), are summarized in table 1. Herein, some further constraints are needed to have a unique classification of order 1 zeros. For example, $(a, b, c) = (c + \frac{1}{2}, c + 1, c)$ is seen to be a solution for the condition (IV), however it follows from the Regge symbol that this corresponds to a trivial zero and hence has to be excluded. Similarly, for those solutions of condition (V) with $a = c$ the Regge symbol can be transformed in such a way that it actually corresponds to a zero of type (I).

The $3j$ coefficients of order 2 are obtained with recurrence relations involving $3j$ coefficients of recursion order 0 and 1 for the two other members in (14). More generally, the $3j$ coefficients of order m are obtained with recurrence relations involving $3j$ coefficients of order $m - 2$ and $m - 1$ for the two other members. In order to characterize the order m in another way, consider the Regge symbol associated with a $3j$ symbol. In this Regge symbol, perform a transformation bringing the two rows or two columns with minimum absolute difference (this is the sum of the absolute values of the differences member by member) to the last two rows. Then, in the correspondence (2), one has:

(iv) if α, β and γ are all integers

$$m = \max\{|\alpha|, |\beta|, |\gamma|\} \qquad \text{if } J \text{ is even}$$

$$m = \max\{|\alpha|, |\beta|, |\gamma|\} - 1 \qquad \text{if } J \text{ is odd}$$

(v) if α, β and γ are not all integers

$$m = \lfloor \max\{|\alpha|, |\beta|, |\gamma|\} \rfloor$$

where $\lfloor x \rfloor$ stands for the integer part of x.

Note that this definition gives the order $m = -1$ for the trivial zeros.

A complete classification for $3j$ coefficients of order 2 and 3 has also been obtained. For order $m = 2$ there are 12 types and for $m = 3$ there are 17 types. Rather than give all these expressions here, we shall only discuss a few typical examples in section 6. The reader interested in the explicit conditions and solutions is referred to a separate report (Raynal and Van der Jeugt 1993).

5. Parametrization of zeros of order 1

The six types of $3j$ coefficients of order 1 have been labelled (I)–(VI) in table 1. The corresponding expressions in the fourth column of table 1 give the conditions under which a $3j$ coefficient of order 1 is *non-trivially* zero. These six conditions are quadratic in the angular momentum quantum numbers a, b and c and, in fact, they are quite easy to solve explicitly. As a consequence, a complete parametrization of zeros of order $m = 1$ is obtained in this section.

Consider first the $3j$ coefficients of order 1 and of type (I). Using

$$K = a(a + 1) - b(b + 1) - c(c + 1) \tag{23}$$

the condition is $K = 0$, which can also be rewritten as $(a - b)(a + b + 1) = c(c + 1)$. Hence there exist two integers $p < q$ without common divisor such that

$$a - b : c + 1 : p = c : a + b + 1 : q.$$

Since $(a-b)q = pc$, c is a multiple of q, say $c = Gq$. Substituting this in $(c+1)q = (a+b+1)p$, it follows that $Gq+1$ must be a multiple of p, say $Gq+1 = gp$. Then the two relations yield $a = \frac{1}{2}(pG+qg-1)$ and $b = \frac{1}{2}(qg-pG-1)$. To parametrize these solutions, let G_1 and g_1 be the smallest (non-negative) values of G and g satisfying $Gq+1 = gp$, and let $G = G_1 + x$ and $g = g_1 + y$. It then follows from $Gq+1 = gp$ that $xq = yp$, so that $x = rp$ and $y = rq$ for some non-negative integer r.

To summarize, the zeros of type (I) can be parametrized as follows. Let $p < q$ be positive integers without common divisor, let G_1 be the smallest non-negative number for which $G_1 q + 1$ is a multiple of p, and put $g_1 = (G_1 q + 1)/p$, then

$$\begin{aligned} &a = \tfrac{1}{2}[(p^2+q^2)r + qg_1 + pG_1 - 1] \qquad b = \tfrac{1}{2}[(q^2-p^2)r + qg_1 - pG_1 - 1] \\ &c = pqr + qG_1 \qquad J = (p+q)(qr+g_1) - 2 \qquad n = (q-p)(pr+G_1) \end{aligned} \tag{24}$$

is a complete parametrization, with r any non-negative integer.

Another solution is found by exchanging c and $c+1$. Then there are positive integers $p < q$ without common divisor such that $q(a-b) = p(c+1)$ and $p(a+b+1) = qc$, and one obtains the following parametrization of zeros of type (I):

$$\begin{aligned} &a = \tfrac{1}{2}[(p^2+q^2)r + qh_1 + pH_1 - 1] \qquad b = \tfrac{1}{2}[(q^2-p^2)r + qh_1 - pH_1 - 1] \\ &c = pqr + ph_1 \qquad J = (p+q)(qr+h_1) - 1 \qquad n = (q-p)(pr+H_1) - 1 \end{aligned} \tag{25}$$

where r is any non-negative integer, and H_1 and h_1 are the smallest values satisfying $Hq = hp + 1$. This shows that every zero lies on two distinct infinite sequences. Indeed, all zeros found by means (25) for some (p, q, r) value also appear in the zeros generated by (24) for some different values (p', q', r'). As a complete parametrization of the zeros of type (I), it is sufficient to use only (24).

Such an analysis can be performed for all the cases, giving rise to a complete parametrization of the zeros of $3j$ coefficients of order 1. This method works here for all cases since the corresponding condition can always be rewritten as a quadratic multiplicative Diophantine equation of the form $x_1 x_2 = u_1 u_2$; such equations have also been studied by Bell (1933) and Srinivasa Rao *et al* (1992).

The zeros of order 1 fall apart into six types, each type providing two infinite sequences of solutions. Since the goal is to parametrize the solutions completely, we give only one of the two sequences here.

In order to give the parametrizations explicitly, we define the following numbers. Let p and q be positive numbers without common divisor, with $p < q$. For k a positive integer (we shall only need the cases $k = 1, 2, 3$), two non-negative numbers G_k and g_k are determined: G_k is the smallest number for which $G_k q + k$ is a multiple of p and g_k follows from $g_k p = G_k q + k$. One can verify that $g_k = k g_1 \bmod q$ and $G_k = k G_1 \bmod p$. We now summarize the parametrizations for the six different types.

For relation (I) a parametrization has been given by (24). The constraint $b \geqslant c$ corresponds to the limitation $q \geqslant (1+\sqrt{2})p$. The parity of J excludes q even and the necessity to obtain integer values for a and b also excludes p even, so p and q should be odd, with any value of r starting from 0 if $p \neq 1$ and from 1 if $p = 1$ because $c = 0$ for $r = 0$ in this case.

The relation (II) can be written as $(a-b)(a+b+1) = (c-1)(c+2)$, that is $q(a-b) = p(c-1)$ and $p(a+b+1) = q(c+2)$, giving the solution

$$\begin{aligned} &a = \tfrac{1}{2}[(p^2+q^2)r + qg_3 + pG_3 - 1] \qquad b = \tfrac{1}{2}[(q^2-p^2)r + qg_3 - pG_3 - 1] \\ &c = pqr + qG_3 + 1 \qquad J = (p+q)(qr+g_3) - 3 \qquad n = (q-p)(pr+G_3) + 1. \end{aligned} \tag{26}$$

The condition $b > c$ gives the limitation already obtained for (I). The parity of J excludes q even and the necessity to obtain integer values for a and b also excludes p even, so p and q should be odd, with any value of r starting from 0 if $p \neq 1, 3$ and from 1 if $p = 1$ or $p = 3$ because $c = 0$ for $r = 0$ in this case.

Relation (III) can be written as the following multiplicative equation:

$$(2a - 2b - c - \tfrac{1}{2})(2a + 2b + c + \tfrac{5}{2}) = 3(c + \tfrac{3}{2})(c - \tfrac{1}{2})$$

that is $q(2a - 2b - c - \frac{1}{2}) = p(c - \frac{1}{2})$ and $p(2a + 2b + c + \frac{5}{2}) = 3q(c + \frac{3}{2})$, giving the solution

$$\begin{aligned}
&a = \tfrac{1}{4}[(3q^2 + p^2)r + 3qg_2 + pG_2 - 2] \\
&b = \tfrac{1}{4}[(q - p)(3q + p)r + (3q - 2p)g_2 - pG_2] \qquad c = pqr + qG_2 + \tfrac{1}{2} \\
&J = \tfrac{1}{2}(3q + p)(qr + g_2) - 2 \qquad n = \tfrac{1}{2}(q - p)(pr + G_2)
\end{aligned} \tag{27}$$

where p and q are not multiples of 3. The constraint $b > c$ corresponds to the limitation $q > (1 + \sqrt{4/3})p$. Here, q and p must be odd; if $q - p$ is a multiple of 4, all the values of r are allowed; if this value is twice an odd number, r is restricted to the parity of g_2.

Relation (IV) can be rewritten as

$$(2a - 2b + c + \tfrac{1}{2})(2a + 2b - c + \tfrac{3}{2}) = 3(c + \tfrac{3}{2})(c - \tfrac{1}{2})$$

that is $q(2a - 2b + c + \frac{1}{2}) = p(c - \frac{1}{2})$ and $p(2a + 2b - c + \frac{3}{2}) = 3q(c + \frac{3}{2})$, giving the solution

$$\begin{aligned}
&a = \tfrac{1}{4}[(3q^2 + p^2)r + 3qg_2 + pG_2 - 2] \\
&b = \tfrac{1}{4}[(q + p)(3q - p)r + (3q + 2p)g_2 - qG_2 - 4] \\
&c = pqr + qG_2 + \tfrac{1}{2} \qquad J = \tfrac{3}{2}(q + p)(qr + g_2) + 1 \qquad n = \tfrac{1}{2}(q + p)(pr + G_2).
\end{aligned} \tag{28}$$

where p and q are not multiples of 3. The constraint $b > c$ corresponds to the limitation $q > p$. Here again, q and p must be odd; if $q + p$ is a multiple of 4, all the values of r are allowed; if this value is twice an odd number, r is restricted to the parity of g_2.

Relation (V) can be written as $(a + \frac{1}{2})(a + c + 1) = b(b + 1)$, that is $q(a + \frac{1}{2}) = pb$ and $p(a + c + 1) = q(b + 1)$, giving the solution

$$\begin{aligned}
&a = p^2r + pG_1 - \tfrac{1}{2} \qquad b = pqr + qG_1 \qquad c = (q^2 - p^2)r + qg_1 - pG_1 - \tfrac{1}{2} \\
&J = (p + q)(qr + g_1) - 2 \qquad n = (q - p)(qr + g_1).
\end{aligned} \tag{29}$$

The triangular relation $a + b \leqslant c$ implies $q \leqslant 2p$. The parity conditions imply that q is odd, and if p is even then $r + g_1$ should be odd.

Relation (VI) can be written $(a + \frac{1}{2})(a - c) = b(b + 1)$, that is $q(a - c) = pb$ and $p(a + \frac{1}{2}) = q(b + 1)$, giving the solution

$$\begin{aligned}
&a = q^2r + qg_1 - \tfrac{1}{2} \qquad b = pqr + qG_1 \qquad c = (q^2 - p^2)r + qg_1 - pG_1 - \tfrac{1}{2} \\
&J = (2q - p)[(q + p)r + g_1 + G_1] \qquad n = (q - p)(pr + G_1).
\end{aligned} \tag{30}$$

There is no limitation other than $q > p$. The parity conditions imply that p is odd and, if q is even, then $r + G_1$ should be odd.

All the zeros of order 1 of the $3j$ coefficients are given by the above formulae as functions of the three parameters p, q and r, subject to the restrictions.

6. On zeros of order 2 and 3

In the previous section, we have shown that the equations for the zeros of order 1 can easily be solved. The equations for zeros of order $m > 1$ are definitely more difficult to tackle.

There are 12 types of zeros with order $m = 2$, and 17 types of zeros with order $m = 3$. These types are summarized in tables 2 and 3, respectively. For $m = 2$ and 3, we do not explicitly give the corresponding Diophantine equation here; this can be found elsewhere (Raynal and Van der Jeugt 1993). However, we do include the degree of this equation, and the number of solutions for $J \leqslant 300$ and $J \leqslant 3000$.

Table 2. The types of $3j$ coefficients $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ of order 2, with $J = a + b + c$. The corresponding vanishing condition is not given explicitly. The last three columns show the degree of this equation, and its number of solutions for $J \leqslant 300$ and for $J \leqslant 3000$.

Type	α	β	γ	J	Degree	$\leqslant 300$	$\leqslant 3000$
2.1	0	2	-2	Even	4	14	19
2.2	1	1	-2	Even	4	1	2
2.3/2.4	0	$\frac{5}{2}$	$-\frac{5}{2}$	Even/odd	4	6/12	10/17
2.5/2.6	1	$\frac{3}{2}$	$-\frac{5}{2}$	Even/odd	4	2/3	7/15
2.7/2.8	$\frac{1}{2}$	$\frac{3}{2}$	-2	Even/odd	3	7/25	31/214
2.9/2.10	$\frac{1}{2}$	2	$-\frac{5}{2}$	Even/odd	4	6/16	13/30
2.11	0	3	-3	Odd	4	8	12
2.12	1	2	-3	Odd	4	0	2

Table 3. The types of $3j$ coefficients $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ of order 3, with $J = a + b + c$. The corresponding vanishing condition is not given explicitly. The last three columns show the degree of this equation, and its number of solutions for $J \leqslant 300$ and for $J \leqslant 3000$.

Type	α	β	γ	J	Degree	$\leqslant 300$	$\leqslant 3000$
3.1	0	3	-3	Even	6	8	10
3.2	1	2	-3	Even	6	4	4
3.3/3.4	0	$\frac{7}{2}$	$-\frac{7}{2}$	Even/odd	6	7/11	12/14
3.5/3.6	1	$\frac{5}{2}$	$-\frac{7}{2}$	Even/odd	6	0/1	0/3
3.7/3.8	$\frac{1}{2}$	$\frac{5}{2}$	-3	Even/odd	5	6/18	8/25
3.9/3.10	$\frac{3}{2}$	$\frac{3}{2}$	-3	Even/odd	4	7/1	17/3
3.11/3.12	$\frac{1}{2}$	3	$-\frac{7}{2}$	Even/odd	6	9/21	13/34
3.13/3.14	$\frac{3}{2}$	2	$-\frac{7}{2}$	Even/odd	6	4/3	5/4
2.15	0	4	-4	Odd	6	6	6
2.16	1	3	-4	Odd	6	0	0
2.17	2	2	-4	Odd	4	2	8

The actual Diophantine equations can be found using recurrence relation (14). For example, setting $(\alpha, \beta, \gamma) = (0, 1, -1)$ in (14) for J even yields, after some algebraic manipulations,

$$K(K+2) - 2b(b+1)c(c+1) = 0 \Rightarrow \begin{pmatrix} a & b & c \\ 0 & 2 & -2 \end{pmatrix} = 0 \qquad (31)$$

where K is given in (23). This is the equation for zeros of type (2.1); the remaining equations are found similarly. Just as in the case of order 1, there are certain constraints. For example, for type (2.1) described in (31), $c \neq a$ since for $c = a$ we have

$$\begin{pmatrix} a & b & a \\ 0 & 2 & -2 \end{pmatrix} = \begin{pmatrix} b & a+1 & a-1 \\ 0 & 1 & -1 \end{pmatrix} \tag{32}$$

and it would correspond to a zero of order 1, of type (I). There are a number of other constraints which we do not list here.

In contrast to the situation of order 1, we have not succeeded in giving all solutions for the zeros of order 2, except for zeros of type (2.7) and (2.8). For the other types except (2.9), we can always give infinite sequences of solutions, however these do not produce all solutions. For example, we cannot solve the equation of (31) completely, but the following describes two infinite sequences of twin solutions:

$$a = (Y^2 \pm Y + 2)/2 \qquad b = (Y^2 \mp Y)/2 \qquad c = (X-1)/2$$

where X and Y are two integers satisfying the following Pell equation

$$X^2 - 8Y^2 = 17. \tag{33}$$

If (X_n, Y_n) satisfies (33), then so does (X_{n+1}, Y_{n+1}) with

$$X_{n+1} = 3X_n + 8Y_n \qquad Y_{n+1} = X_n + 3Y_n.$$

Since for any value of (X_n, Y_n) there are two zeros, one can generate two sequences of twin zeros with initial starting values $(X_0, Y_0) = (5, 1)$ and $(X_0, Y_0) = (7, 2)$. This shows that there are an infinite number of zeros of order $m = 2$ of type (2.1).

The zeros of type (2.7) and (2.8) deserve further attention, since in this case a complete solution can be given. We describe here only the ones of type (2.7); (2.8) is similar. For (2.7), J even, the Diophantine equation is

$$(a+\tfrac{1}{2})(c-1)(c+2) = (a+b+2)(a+b)(a-b) \Rightarrow \begin{pmatrix} a & b & c \\ \frac{1}{2} & \frac{3}{2} & -2 \end{pmatrix} = 0. \tag{34}$$

Let $p > q$ be two integers without a common divisor. Using $a + \frac{1}{2} = pY$, $b + \frac{1}{2} = qY$ and $2c + 1 = X/p$, the condition can be rewritten as

$$X^2 - 4p(p-q)(p+q)^2Y^2 = p(5p+4q). \tag{35}$$

For every p and q this is a Pell equation in X and Y, and sequences of solutions can be obtained as for the Pell equation (33). Every zero of type (2.7) belongs to such a sequence, for some p and q.

The situation for zeros of order 3 is similar to that of order 2. Here, none of the 17 types has been solved completely, but for some types we were able to show that they possess infinite sequences of solutions, related to some Pell equation. We give one example here. For zeros of type (3.9), with J even, the equation can be written in the following form:

$$\begin{aligned}[][a(a+1) - b(b+1)](K+1) - (b-\tfrac{1}{2})(b+\tfrac{3}{2})(c-1)(c+2) + c(c+1)(a-b)(b+\tfrac{1}{2}) \\ = 0 \Rightarrow \begin{pmatrix} a & b & c \\ \frac{3}{2} & \frac{3}{2} & -3 \end{pmatrix} = 0. \end{aligned} \tag{36}$$

Table 4. The number of zeros of degree n for $3j$ coefficients as a function of $J = a + b + c$ is the sum of the two figures given in this table. The second figure is the number of zeros of order $m \leqslant 3$.

J	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
1–30	13, 19	4, 9	0, 0	0, 2	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
31–60	138, 31	34, 26	13, 8	7, 8	3, 2	0, 4	0, 1	0, 6	0, 3	0, 0	0, 0
61–90	334, 35	65, 19	29, 6	15, 8	5, 3	0, 8	1, 1	0, 9	0, 1	0, 3	0, 2
91–120	552, 32	128, 17	39, 5	17, 5	4, 0	2, 5	0, 0	2, 7	0, 1	0, 6	0, 0
121–150	725, 32	101, 13	37, 0	9, 1	2, 0	1, 3	1, 1	0, 4	1, 1	0, 2	0, 0
151–180	1034, 32	150, 20	47, 2	17, 1	7, 0	0, 1	1, 0	0, 3	0, 0	0, 4	0, 0
181–210	1336, 32	182, 10	40, 1	20, 1	6, 0	2, 2	4, 0	0, 4	0, 0	0, 4	1, 1
211–240	1513, 32	184, 8	43, 0	7, 2	3, 0	0, 1	0, 0	0, 0	0, 0	0, 2	0, 0
241–270	1785, 32	147, 4	46, 1	12, 0	1, 1	1, 0	0, 1	0, 0	0, 0	0, 0	0, 0
271–300	2239, 35	283, 5	58, 0	9, 0	6, 0	2, 1	1, 0	0, 1	0, 0	0, 5	0, 0
301–331	2375, 32	176, 3	35, 0	12, 0	3, 0	0, 1	0, 0	0, 0	0, 0	0, 1	0, 0
331–360	2726, 32	198, 2	51, 0	8, 1	2, 0	2, 0	1, 1	0, 0	0, 0	0, 0	0, 0
361–390	2995, 32	250, 5	36, 0	17, 0	1, 0	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0
391–420	3425, 32	247, 2	52, 0	6, 0	0, 0	0, 0	0, 0	0, 1	0, 0	0, 1	0, 0
421–450	3658, 32	236, 0	40, 0	7, 0	2, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
451–480	3985, 32	276, 0	47, 0	8, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
481–510	4228, 35	263, 5	56, 0	14, 0	5, 0	2, 0	0, 0	0, 1	0, 0	0, 0	0, 0
511–540	4716, 32	282, 1	50, 0	14, 0	7, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 0
541–570	4770, 32	281, 0	56, 0	14, 0	4, 0	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0
571–600	5500, 32	266, 1	46, 0	7, 0	2, 0	0, 1	0, 0	0, 0	0, 0	0, 1	0, 0
601–630	5790, 32	320, 0	47, 0	8, 1	1, 0	0, 0	0, 0	0, 0	1, 0	0, 1	0, 0
631–660	5852, 32	281, 0	42, 0	5, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
661–690	6289, 32	279, 1	42, 0	10, 0	3, 0	0, 0	1, 0	0, 0	0, 0	1, 0	0, 0
691–720	6895, 35	324, 1	51, 0	6, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
721–750	6915, 32	304, 0	56, 0	4, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 1	0, 0
751–780	7536, 32	285, 0	40, 0	5, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 1	0, 0
781–810	7558, 32	299, 2	61, 0	15, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
811–840	8585, 32	323, 0	39, 0	4, 0	1, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 0
841–870	8314, 32	356, 0	41, 0	4, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
871–900	9088, 32	343, 0	43, 0	3, 0	1, 0	0, 0	0, 0	0, 0	0, 0	0, 1	0, 0

A sequence of solutions is given by

$$a = (2X+1)/6 \qquad b = (X+2)/6 \qquad c = (Y-1)/2$$

where X and Y are two integers (with $X \bmod 6 = 1$ and Y odd), satisfying the following Pell equation

$$X^2 - 3Y^2 = -26. \tag{37}$$

If (X_n, Y_n) satisfies (37), then so does (X_{n+1}, Y_{n+1}) with

$$X_{n+1} = 7X_n + 12Y_n \qquad Y_{n+1} = 4X_n + 7Y_n$$

and, moreover, $X_{n+1} \bmod 6 = 1$ and Y_{n+1} is odd. Here, one can generate two sequences of solutions with initial starting values $(X_0, Y_0) = (43, 25)$ and $(X_0, Y_0) = (109, 63)$.

7. Further results and comments

The zeros of low degree $n \leqslant 12$ have been found up to $J = 900$ and their number is given in table 4. From this table, one can indeed verify that the majority of zeros of high degree n seem to have low order m. In this section, we shall discuss some further results which follow from the table and from explicit lists of structural zeros.

The parameters x, y, z and t of (3) will be used here. However, for the convenience of discussion, we shall not use the restrictions (4) which were introduced to take into account symmetries: x and t, y and z, and also the couples (x, t) and (y, z), may be interchanged.

The zeros of degree $n = 1$ are obtained by the equation $xt = yz$. They can be written as

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} pd & qd & (p+q)d-1 \\ p\delta & q\delta & -(p+q)\delta \end{pmatrix} \tag{38}$$

where p and q are two integers without a common divisor, d is an integer or half-integer larger than 1 and $|\delta| < d$ with $d + \delta$ integer. The value $d = \frac{1}{2}$ gives only $3j$ coefficients of degree 0, which cannot vanish; the value $d = 1, \delta = 0$ gives a trivial zero. Note that the value $J + 1 = a + b + c + 1 = 2(p + q)d$ must be divisible by $2d > 2$ and thus cannot be twice a prime number. From $J = 8$ onwards, there exist zeros for all the values of $J + 1$ which are not a prime number nor twice a prime number. For $J \leqslant 900$, there are 121 827 of them. The maximum number of zeros obtained for a single value of J is 1048 for $J = 839$.

There are 7021 zeros of degree $n = 2$ for $J \leqslant 900$. For a fixed value of J, there can be many of them: up to 73 for $J = 594$ and for $J = 714$. The zeros of degree 2 have been studied by Louck and Stein (1987) who obtained infinite families with fixed values for two parameters, one of each set (x, t) and (y, z).

There are 1306 zeros of degree $n = 3$ for $J \leqslant 900$, for 441 different values of J. From the list of zeros, we noticed that 422 of them satisfy the relation $x(t-1) = (y-1)(z-1)$. Introducing two integers $p > q$ without a common divisor and writing $x = p(y-1)/q$, $z = 1 + p(t-1)/q$, the condition for a zero of degree $n = 3$ reduces to the simple equation $(p + 2q)t = (p - q)y$. Hence we put

$$\frac{p+2q}{p-q} = \frac{g}{h}$$

where $g > h$ has no common divisor. Then $t = Sh$ and $y = Sg$ for some integer S; substituting this in the expressions for x and z yields $x = p(Sg - 1)/q$ and $z = 1 + p(Sh - 1)/q$. Since x and z should also be integers, let s be the smallest integer such that q is a divisor both of $sg - 1$ and of $sh - 1$. Then $S = s + rq$, with r any integer, gives the most general solution:

$$x = p((sg-1)/q) + r(gp) \qquad y = sg + r(gq)$$
$$z = 1 + p((sh-1)/q) + r(hp) \qquad t = sh + r(hq).$$

This is a parametrization for a large class of degree 3 zeros, and shows that there are infinite sequences of zeros with $n = 3$.

There are 314 zeros of degree $n = 4$ for $J \leqslant 900$, 30 of which have order $m \leqslant 3$. For degree 4 we were also able to deduce two infinite sequences of solutions from inspection of the explicit list of zeros. We give one example here:

$$J = \tfrac{1}{2}(d+2)(3d+11) \qquad x = \tfrac{1}{2}(d+1)(3d+2) \qquad y = 3d+5 \qquad z = 3d \qquad t = 9$$

where $d \geqslant 2$ is an integer.

There are 78 zeros of degree $n = 5$ for $J \leqslant 900$, of which only six have order $m \leqslant 3$. As the degree n increases, one can see from table 4 that the number of zeros with order $m > 3$ decreases very rapidly. For $J \leqslant 900$, this number is 15, 10, 3, 2, 1 and 1 for zeros of degree $n = 6, 7, 8, 9, 10$ and 11 respectively. The corresponding zeros are given explicitly in table 5. For $n \geqslant 12$, no zeros with $m > 3$ have been encountered for $J \leqslant 900$.

Table 5. Zeros of $3j$ coefficients of order $m > 3$ and degree $6 \leqslant n \leqslant 11$ with $J \leqslant 900$. If $n > m$ the $3j$ coefficient corresponding to the Regge symbol (3) is given in the last columns. If $n < m$, it is replaced by the $3j$ coefficients with the smallest magnetic quantum numbers. If $m = n$, the two coefficients are given.

J	m	n	x	y	z	t	a	b	c	α	β	γ
97	5	6	35	25	17	26	$\frac{91}{2}$	$\frac{43}{2}$	30	$\frac{1}{2}$	$\frac{9}{2}$	-5
120	7	6	38	32	21	35	$\frac{59}{2}$	$\frac{67}{2}$	57	$\frac{17}{2}$	$-\frac{3}{2}$	-7
134	5	6	46	35	24	35	64	$\frac{81}{2}$	$\frac{59}{2}$	0	$\frac{11}{2}$	$-\frac{11}{2}$
189	47	6	132	43	13	7	$\frac{145}{2}$	25	$\frac{183}{2}$	$\frac{119}{2}$	18	$-\frac{155}{2}$
200	35	6	119	60	19	8	69	34	97	50	26	-76
259	87	6	189	36	31	9	110	$\frac{45}{2}$	$\frac{253}{2}$	79	$\frac{27}{2}$	$-\frac{185}{2}$
287	65	6	186	63	26	18	106	$\frac{81}{2}$	$\frac{281}{2}$	80	$\frac{45}{2}$	$-\frac{205}{2}$
288	10	6	95	78	62	59	$\frac{157}{2}$	$\frac{137}{2}$	141	$\frac{33}{2}$	$\frac{19}{2}$	-26
338	54	6	189	95	37	23	113	59	166	76	36	-112
347	8	6	136	121	40	56	88	$\frac{177}{2}$	$\frac{341}{2}$	48	$\frac{65}{2}$	$-\frac{161}{2}$
494	41	6	246	164	44	46	145	105	244	101	59	-160
494	40	6	261	185	29	25	145	105	244	116	80	-196
531	104	6	288	130	85	34	$\frac{373}{2}$	82	$\frac{525}{2}$	$\frac{203}{2}$	48	$-\frac{299}{2}$
560	125	6	378	143	30	15	204	79	277	174	64	-238
832	249	6	630	157	38	13	334	85	413	296	72	-368
89	9	7	39	22	18	17	$\frac{57}{2}$	$\frac{39}{2}$	41	$\frac{21}{2}$	$\frac{5}{2}$	-13
145	7	7	52	39	23	38	$\frac{75}{2}$	$\frac{77}{2}$	69	$\frac{29}{2}$	$\frac{1}{2}$	-15
							69	$\frac{91}{2}$	$\frac{61}{2}$	1	$\frac{13}{2}$	$-\frac{15}{2}$
177	7	7	69	54	23	38	46	46	85	23	8	-31
							85	$\frac{123}{2}$	$\frac{61}{2}$	0	$\frac{15}{2}$	$-\frac{15}{2}$
186	29	7	117	58	9	9	63	$\frac{67}{2}$	$\frac{179}{2}$	54	$\frac{49}{2}$	$-\frac{157}{2}$
187	4	7	66	57	32	39	90	$\frac{71}{2}$	$\frac{123}{2}$	1	$\frac{7}{2}$	$-\frac{9}{2}$
208	62	7	156	34	14	11	85	$\frac{45}{2}$	$\frac{201}{2}$	71	$\frac{23}{2}$	$-\frac{165}{2}$
209	13	7	64	61	32	59	48	60	101	16	1	-17
289	77	7	188	48	38	22	113	35	141	75	13	-88
357	8	7	154	136	28	46	91	91	175	63	45	-108
665	164	7	475	158	25	14	250	86	329	225	72	-297
115	19	8	66	28	15	14	$\frac{81}{2}$	21	$\frac{107}{2}$	$\frac{51}{2}$	7	$-\frac{65}{2}$
116	19	8	65	32	16	11	$\frac{81}{2}$	$\frac{43}{2}$	54	$\frac{49}{2}$	$\frac{21}{2}$	-35
361	13	8	118	92	66	93	92	$\frac{185}{2}$	$\frac{353}{2}$	26	$-\frac{1}{2}$	$-\frac{51}{2}$
146	6	9	52	40	25	38	$\frac{137}{2}$	46	$\frac{63}{2}$	$\frac{1}{2}$	6	$-\frac{13}{2}$
611	16	9	215	215	78	112	$\frac{293}{2}$	$\frac{327}{2}$	301	$\frac{137}{2}$	$\frac{103}{2}$	-120
667	163	10	477	159	25	16	251	$\frac{175}{2}$	$\frac{657}{2}$	226	$\frac{143}{2}$	$-\frac{595}{2}$
188	4	11	65	62	32	40	$\frac{127}{2}$	$\frac{177}{2}$	36	$\frac{3}{2}$	$\frac{5}{2}$	-4

8. Conclusions

We have studied the structural zeros of $3j$ coefficients. In previous works, these zeros have been studied and classified according to their degree n. An extended computer search for such zeros indicated that a new parameter, the recurrence order m, could be helpful in classifying the structural zeros. This new parameter was defined, and the equations for zeros of order $m = 1$ were completely solved. Zeros of order 2 and 3 were classified, and infinite sequences of solutions were presented; these solutions are not, in general, complete. Regarding the degree of non-trivial zeros, it was known that there were an infinite number of degree 1 and 2. Here, the explicit computer list of zeros was helpful in finding infinite sequences of zeros of degree 3 and 4. It is not known whether the number of zeros of degree n, where $n > 4$, is finite or infinite.

Table 4 presents the frequency of zeros for J-intervals emphasizing the 'degree versus order' theme of this paper. If we consider the zeros with $n > m$ and arrange them according to increasing values of m, it appears that there are no zeros of high order. In the range of this search, there is only one zero of order $m = 6$ with degree $n = 9$, there are two zeros of order $m = 5$ with degree $n = 6$ and five zeros of order $m = 4$: three with $n = 5$, one for $n = 7$ and one for $n = 11$. Conversely, if we consider the set of zeros with $n \leqslant m$ and arrange them according to n, then it appears that there are no zeros of high degree. In this set (for $J \leqslant 900$), we found only one zero with $n = 10$, two with $n = 9$, three with $n = 8$, and a small number with $n = 7$ and $n = 6$; all of these are given in table 5.

The reader interested in a detailed analysis of order 2 and 3 zeros, and in a further classification of zeros of degree 2, 3 and 4, is referred to a separate report (Raynal and Van der Jeugt 1993). For $6j$ coefficients, a similar analysis is now being performed and we hope this will be the subject of a future paper.

All the manipulations of algebraic expressions have been done using the AMP language written by Drouffe (1982).

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